Problem 1: Using duality to prove optimality

Basic Idea

- Let D be the dual linear program of P, x^* be a feasible solution for P, and y^* be a feasible solution for D. According to the complementary slackness theorem, the following statements are equivalent.
  - x^* is optimal for P and y^* is optimal for D.
  - For every constraint i in P, x^* satiesfies the ith constraint with equality, i.e. without slack, or $y_i^* = 0$ in D. For every constraint j in D, y^* satisfies the jth constraint with equality, or $x_j^* = 0$ in P.
- x^* satisfies the constraints 1, 2, 5 with equality in P.
- $x_3^* \neq 0 \wedge x_4^* \neq 0 \wedge x_6^* \neq 0$, therefore we replace the 3rd, 4th, and 6th constraint in D with equality constraints.
- We set $y_3^* = y_4^* = y_6^* = 0$ in D and show that the the problem is still feasible. This proves that x^* and y^* are optimal.

Proof

We check which constraints in P are satisfied by x^* with equality, i.e. without slack.

$$\begin{aligned}
x_1 - 4x_3 + 3x_4 + x_5 + x_6 &= 1 && \text{no slack} \\
5x_1 + 3x_2 + x_3 - 5x_5 + 3x_6 &= 4 && \text{no slack} \\
4x_1 + 5x_2 - 3x_3 + 3x_4 - 4x_5 + x_6 &= \frac{7}{2} < 4 && \text{slack} \\
-x_2 + 2x_4 + x_5 - 5x_6 &= \frac{9}{2} < 5 && \text{slack} \\
-2x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_5 &= 7 && \text{no slack} \\
2x_1 - 3x_2 + 2x_3 - x_4 + 4x_5 + 5x_6 &= 4 < 5 && \text{slack}
\end{aligned}$$

We write down the dual linear program D.

$$\begin{aligned}
\text{minimize} \quad & y_1 + 4y_2 + 4y_3 + 5y + 4 + 7y_5 + 5y_6 \\
\text{subject to} \quad & y_1 + 5y_2 + 4y_3 - y_4 - 2y_5 + 2y_6 \geq 4 \\
& 3y_2 + 5y_2 - y_4 + y_5 - 3y_6 \geq 5 \\
& -4y_1 + y_2 - 3y_3 + y_5 + 2y_6 \geq 1 \\
& 3y_1 + 3y_3 + 2y_4 + y_5 - y_6 \geq 3 \\
& y_1 - 5y_2 - 4y_3 + y_4 + 2y_5 + 4y_6 \geq -5 \\
& y_1 + 3y_2 + y_3 - y_4 + 2y_5 + 5y_6 \geq 8 \\
& y_1, y_2, y_3, y_4, y_5 \geq 0
\end{aligned}$$

If x^* is an optimal solution if D has a feasible solution with $y_3 = y_4 = y_6 = 0$ and equality constraints i for $x_i^* \neq 0$.

$$\begin{aligned}
\text{minimize} \quad & y_1 + 4y_2 + 7y_5 \\
\text{subject to} \quad & y_1 + 5y_2 - 2y_5 \geq 4 \\
& 3y_2 + y_5 \geq 5 \\
& -4y_1 + y_2 + y_5 = 1 \\
& 3y_1 + y_5 = 3 \\
& y_1 - 5y_2 + 2y_5 \geq -5 \\
& y_1 + 3y_2 + 2y_5 = 8 \\
& y_1, y_2, y_5 \geq 0
\end{aligned}$$

This linear system contains three equations with three equalities. We can solve it and get $y_1 = \frac{1}{2}$ and $y_2 = y_5 = \frac{3}{2}$. These values also satisfy the inequalities. Therefore, x^* is an optimal solution. Also, if we plug in the values for x^* and y^* in the objective function for P and D, we get the same value 17. ■

Problem 2: Simplex Method

We transform the linear program to equational form.

$$\begin{aligned} \text{maximize} \quad & 3x_1 + x_2 \\ \text{subject to} \quad & x_1 - x_2 + x_3 = -1 \\ & -x_1 - x_2 - x_4 = -3 \\ & 2x_1 + x_2 + x_5 = 4 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Auxiliary linear program (Simplex Phase 1)

We solve an auxiliary linear program to find an initial basic feasible solution. We multiply the first and the second constraint with -1 and add a slack variable for every constraint.

$$\begin{aligned} \text{maximize} \quad & -x_6 - x_7 - x_8 \\ \text{subject to} \quad & -x_1 + x_2 - x_3 + x_6 = 1 \\ & x_1 + x_2 - x_4 + x_7 = 3 \\ & 2x_1 + x_2 + x_5 + x_8 = 4 \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0 \end{aligned}$$

We solve the auxiliary linear program with the simplex method. $x_6 = 1$, $x_7 = 3$, $x_8 = 4$, $x_1 = x_2 = x_3 = x_4 = x_5 = 0$ is an initial basic feasible solution.

$$\begin{array}{r|c} x_6 & = 1 + x_1 - x_2 + x_3 \\ x_7 & = 3 - x_1 - x_2 + x_4 \\ x_8 & = 4 - 2x_1 - x_2 - x_5 \\ \hline z & = -8 + 2x_1 + 3x_2 - x_3 - x_4 + x_5 \end{array}$$

- Pivot by x_2
- x_6 $x_2 \leq 1$, x_7 $x_2 \leq 1$, x_8 $x_2 \leq 1$
- New non-basic variable: x_6

$$\begin{array}{r|c} x_2 & = 1 + x_1 + x_3 - x_6 \\ x_7 & = 2 - 2x_1 - x_3 + x_6 + x_4 \\ x_8 & = 3 - 3x_1 - x_3 - x_5 + x_6 \\ \hline z & = -5 + 5x_1 + 2x_3 - 3x_6 - x_4 + x_5 \end{array}$$

- Pivot by x_3
- x_2 unconstr., x_7 $x_3 \leq 2$, x_8 $x_3 \leq 3$
- New non-basic variable: x_7

$$\begin{array}{r|c} x_3 & = 2 - 2x_1 + x_4 + x_6 - x_7 \\ x_2 & = 3 - x_1 + x_4 - x_7 \\ x_8 & = 1 - x_1 - x_4 - x_5 + x_7 \\ \hline z & = -1 + x_1 + x_4 - x_6 - 2x_7 + x_5 \end{array}$$

- Pivot by x_5
- x_3 unconstr., x_2 unconstr., x_8 $x_5 \leq 1$
- New non-basic variable: x_8

$$\begin{array}{c|c} x_5 & = 1 - x_1 - x_4 + x_7 - x_8 \\ x_3 & = 2 - 2x_1 + x_4 + x_6 - x_7 \\ x_2 & = 3 - x_1 + x_4 - x_7 \\ \hline z & = -x_6 - x_7 - x_8 \end{array}$$

- Optimal solution found
- Solution: $x_5 = 1$, $x_3 = 2$, $x_2 = 3$, others 0
- Objective function value 0, thus feasible

Solve linear program (Simplex Phase 2)

We use the solution from the auxiliary linear program as an initial basic feasible solution and run the simplex algorithm. In the auxiliary linear program, the objective function value was 0, therefore, its solution was a basic feasible solution for the actual linear program.

$$\begin{array}{c|c} x_5 & = 1 - x_1 + x_4 \\ x_3 & = 2 - 2x_1 - x_4 \\ x_2 & = 3 - x_1 - x_4 \\ \hline z & = 3 + 2x_1 - x_4 \end{array}$$

- Pivot by x_1
- x_5 $x_1 \leq 1$, x_3 $x_1 \leq 1$, x_2 $x_1 \leq 3$
- New non-basic variable: x_5

$$\begin{array}{c|c} x_1 & = 1 + x_4 - x_5 \\ x_3 & = -3x_4 + 2x_5 \\ x_2 & = 2 - 2x_4 + x_5 \\ \hline z & = 5 + x_4 - 2x_5 \end{array}$$

- Pivot by x_4
- x_1 unconstr., x_3 $x_4 \leq 0$, x_2 $x_4 \leq 1$
- New non-basic variable: x_3

$$\begin{array}{c|c} x_4 & = \frac{2x_5}{3} - \frac{x_3}{3} \\ x_1 & = 1 - \frac{x_5}{3} - \frac{x_3}{3} \\ x_2 & = 2 - \frac{x_5}{3} + \frac{2x_3}{3} \\ \hline z & = 5 - \frac{x_3}{3} - 2x_5 \end{array}$$

- Optimal solution found
- Solution: $x_1 = 1$, $x_2 = 2$, $x_3 = x_4 = x_5 = 0$

$x_1 = 1$, $x_2 = 2$, $x_3 = 0$ is an optimal solution for the linear program, achieving an objective function value 5.

Problem 3: Formulating the dual program

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 + x_3 \\ \text{subject to} \quad & x_1 + x_3 \leq 2 \\ & x_1 + 2x_2 = 2 \\ & x_1 + x_2 + x_3 \leq 3 \\ & x_1, x_3 \leq 0 \end{aligned}$$

$$\begin{aligned} \text{minimize} \quad & 2y_1 + 2y_2 + 3y_3 \\ \text{subject to} \quad & y_1 + y_2 + y_3 \leq 1 \\ & 2y_2 + y_3 = 1 \\ & y_1 + y_3 \leq 1 \\ & y_1, y_3 \geq 0 \end{aligned}$$

Problem 4

A_x and B_x are variables for the amount of raw gasoline of type x in Avgas A and Avgas B, respectively. A and B is the amount of produced Avgas A and Avgas B. R_x is the amount of sold raw gasoline of type x. R is the total amount of sold raw gasoline.

$$\begin{aligned}
\text{maximize} \quad & 6.45A + 5.91B + 4.83R \\
\text{subject to} \quad & 107A_A + 93A_C + 87A_S + 108A_I \geq 100A \\
& 107B_A + 93B_C + 87B_S + 1087B_I \geq 91B \\
& 5A_A + 8A_C + 4A_S + 21A_I \leq 7A \\
& 5B_A + 8B_C + 4B_S + 21B_I \leq 7B \\
& A_A + A_C + A_S + A_I = A \\
& B_A + B_C + B_S + B_I = B \\
& R_A + R_C + R_S + R_I = R \\
& R_A + A_A + B_A = 3814 \\
& R_C + A_C + B_C = 2666 \\
& R_S + A_S + B_S = 4016 \\
& R_I + A_I + B_I = 1300 \\
& A, A_A, A_C, A_S, A_I, B, B_A, B_C, B_S, B_I, R, R_A, R_C, R_S, R_I \geq 0
\end{aligned}$$

Now, we convert the linear program to canonical form.

$$\begin{aligned}
\text{maximize} \quad & 6.45A + 5.91B + 4.83R \\
\text{subject to} \quad & -107A_A - 93A_C - 87A_S - 108A_I + 100A \leq 0 \\
& -107B_A - 93B_C - 87B_S - 1087B_I + 91B \leq 0 \\
& 5A_A + 8A_C + 4A_S + 21A_I - 7A \leq 0 \\
& 5B_A + 8B_C + 4B_S + 21B_I - 7B \leq 0 \\
& A_A + A_C + A_S + A_I - A \leq 0 \\
& -A_A - A_C - A_S - A_I + A \leq 0 \\
& B_A + B_C + B_S + B_I - B \leq 0 \\
& -B_A - B_C - B_S - B_I + B \leq 0 \\
& R_A + R_C + R_S + R_I - R \leq 0 \\
& -R_A - R_C - R_S - R_I + R \leq 0 \\
& R_A + A_A + B_A \leq 3814 \\
& -R_A - A_A - B_A \leq -3814 \\
& R_C + A_C + B_C \leq 2666 \\
& -R_C - A_C - B_C \leq -2666 \\
& R_S + A_S + B_S \leq 4016 \\
& -R_S - A_S - B_S \leq -4016 \\
& R_I + A_I + B_I \leq 1300 \\
& -R_I - A_I - B_I \leq -1300 \\
& A, A_A, A_C, A_S, A_I, B, B_A, B_C, B_S, B_I, R, R_A, R_C, R_S, R_I \geq 0
\end{aligned}$$

Problem 5: Nash equilibrium

Row Player

$$\begin{aligned} \text{maximize} \quad & x_0 \\ \text{subject to} \quad & -x_1 - x_2 - x_0 \geq 0 \\ & x_1 - x_2 + x_3 - x_0 \geq 0 \\ & -x_1 + x_2 + x_3 - x_0 \geq 0 \\ & 2x_1 + x_2 - x_3 - x_0 \geq 0 \\ & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Column Player

$$\begin{aligned} \text{minimize} \quad & y_0 \\ \text{subject to} \quad & -y_1 + y_2 - y_3 + 2y_4 - y_0 \leq 0 \\ & -y_1 - y_2 + y_3 + y_4 - y_0 \leq 0 \\ & y_2 + y_3 - y_4 - y_0 \leq 0 \\ & y_1 + y_2 + y_3 + y_4 = 1 \\ & y_1, y_2, y_3, y_4 \geq 0 \end{aligned}$$

Solving with the Simplex Method (Row Player)

We transform the linear program to equational form by introducing new slack variables and substituting $x_0 = x_4 - x_5$, where $x_4, x_5 \geq 0$.

$$\begin{aligned} \text{maximize} \quad & x_4 - x_5 \\ \text{subject to} \quad & -x_1 - x_2 - x_4 + x_5 - x_6 = 0 \\ & x_1 - x_2 + x_3 - x_4 + x_5 - x_7 = 0 \\ & -x_1 + x_2 + x_3 - x_4 + x_5 - x_8 = 0 \\ & 2x_1 - x_2 - x_3 - x_4 + x_5 - x_9 = 0 \\ & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \geq 0 \end{aligned}$$

Using trail and error, we guess an initial basic feasible solution: $x_1 = x_2 = x_4 = x_9 = 0$, $x_3 = 1$, $x_5 = 1$, $x_8 = 2$, $x_7 = 2$, $x_6 = 1$. We use the simplex method to solve the linear program.

$$\begin{array}{c|l} x_3 & = 1 - x_2 - x_1 \\ x_6 & = 1 - 4x_1 - x_2 + x_9 \\ x_7 & = 2 - 2x_2 - 3x_1 + x_9 \\ x_8 & = 2 - 3x_1 + x_9 \\ x_5 & = 1 - 3x_1 + x_4 + x_9 \\ \hline z & = -1 + 3x_1 - x_9 \end{array}$$

- Pivot by x_1
- x_3 $x_1 \leq 1$, x_6 $x_1 \leq \frac{1}{4}$
- x_7 $x_1 \leq \frac{2}{3}$, x_8 $x_1 \leq \frac{2}{3}$, x_5 $x_1 \leq \frac{1}{3}$
- New non-basic variable: x_6

$$\begin{array}{c|l} x_1 & = \frac{1}{4} - \frac{x_2}{4} + \frac{x_9}{4} - \frac{x_6}{4} \\ x_3 & = \frac{3}{4} - \frac{3x_2}{4} - \frac{x_9}{4} + \frac{x_6}{4} \\ x_7 & = \frac{5}{4} - \frac{5x_2}{4} + \frac{x_9}{4} + \frac{3x_6}{4} \\ x_8 & = \frac{5}{4} + \frac{3x_2}{4} + \frac{x_9}{4} + \frac{3x_6}{4} \\ x_5 & = -\frac{1}{4} - \frac{3x_2}{4} - \frac{x_9}{4} + \frac{3x_6}{4} + x_4 \\ \hline z & = -\frac{1}{4} - \frac{3x_2}{4} - \frac{x_9}{4} - \frac{3x_6}{4} \end{array}$$

- Optimal solution found
- $x_0 = x_4 - x_5 = -\frac{1}{4}$
- $x_1 = \frac{1}{4}$, $x_2 = 0$, $x_3 = \frac{3}{4}$

Solving with the Simplex Method (Column Player)

We transform the linear program to equational form by introducing new slack variables and substituting $y_0 = y_6 - y_5$, where $y_6, y_5 \geq 0$.

$$\begin{aligned} \text{maximize} \quad & y_5 - y_6 \\ \text{subject to} \quad & -y_1 + y_2 - y_3 + 2y_4 + y_5 - y_6 + y_7 = 0 \\ & -y_1 - y_2 + y_3 + y_4 + y_5 - y_6 - y_8 = 0 \\ & y_2 + y_3 - y_4 + y_5 - y_6 + y_9 = 0 \\ & y_1 + y_2 + y_3 + y_4 = 1 \\ & y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9 \geq 0 \end{aligned}$$

Using trail and error, we guess an initial basic feasible solution: $y_2 = y_3 = y_4 = y_5 = y_6 = y_9 = 0$, $y_1 = 1$, $y_8 = 1$, $y_7 = 1$. We use the simplex method to solve the linear program.

$$\begin{array}{c|c} y_1 & = 1 - y_2 - y_3 - y_4 \\ y_6 & = y_2 + y_3 - y_4 + y_5 + y_9 \\ y_8 & = 1 + y_2 - y_3 - 3y_4 + y_9 \\ y_7 & = 1 - y_2 + y_3 - 4y_4 + y_9 \\ \hline z & = -y_2 - y_3 + y_4 - y_9 \end{array}$$

- Pivot by y_4
- y_1 $y_4 \leq 1$, y_6 $y_4 \leq 0$
- y_8 $y_4 \leq \frac{1}{3}$, y_7 $y_4 \leq 1$
- New non-basic variable: y_6

$$\begin{array}{c|c} y_4 & = y_2 + y_3 + y_5 - y_6 + y_9 \\ y_1 & = 1 - 2y_2 - 2y_3 - y_5 + y_6 - y_9 \\ y_8 & = 1 - 2y_2 - 4y_3 - 3y_5 - 3y_6 - 3y_9 \\ y_7 & = 1 - 5y_2 - 3y_3 - 4y_5 - 4y_6 - 4y_9 \\ \hline z & = y_5 - y_6 \end{array}$$

- Pivot by y_5
- y_4 unconstr., y_1 $y_5 \leq 1$
- y_8 $y_5 \leq \frac{1}{3}$, y_7 $y_5 \leq \frac{1}{4}$
- New non-basic variable: y_7

$$\begin{array}{c|c} y_4 & = \frac{1}{4} - \frac{y_2}{4} + \frac{y_3}{4} - y_6 - \frac{y_7}{4} - y_9 \\ y_1 & = \frac{3}{4} - \frac{3y_2}{4} - \frac{5y_3}{4} + y_6 + y_9 + \frac{y_7}{4} \\ y_8 & = \frac{1}{4} + \frac{7y_2}{4} + \frac{y_3}{4} + 3y_6 + 3y_9 + \frac{3y_7}{4} \\ y_5 & = \frac{1}{4} - \frac{5y_2}{4} - \frac{3y_3}{4} - y_6 - y_9 - \frac{y_7}{4} \\ \hline z & = \frac{1}{4} - \frac{5y_2}{4} - \frac{3y_3}{4} - y_6 - y_9 - \frac{y_7}{4} \end{array}$$

- Optimal solution found
- $y_0 = y_6 - y_5 = -\frac{1}{4}$
- $y_1 = \frac{3}{4}$, $y_2 = y_3 = 0$, $y_4 = \frac{1}{4}$

Problem 6: Flow Problem

Linear Program

$$
\begin{aligned}
\text{maximize} \quad & f_{CT} + f_{DT} \\
\text{subject to} \quad & 0 \leq f_{SA} \leq 8 \\
& 0 \leq f_{SB} \leq 10 \\
& 0 \leq f_{AB} \leq 9 \\
& 0 \leq f_{AC} \leq 3 \\
& 0 \leq f_{BD} \leq 6 \\
& 0 \leq f_{AD} \leq 8 \\
& 0 \leq f_{BC} \leq 6 \\
& 0 \leq f_{CD} \leq 5 \\
& 0 \leq f_{CT} \leq 10 \\
& 0 \leq f_{DT} \leq 9 \\
& f_{SA} = f_{AB} + f_{AD} + f_{AC} \\
& f_{SB} + f_{AB} = f_{BC} + f_{BD} \\
& f_{AC} + f_{BC} = f_{CT} + f_{CD} \\
& f_{BD} + f_{AD} + f_{CD} = f_{DT}
\end{aligned}
$$

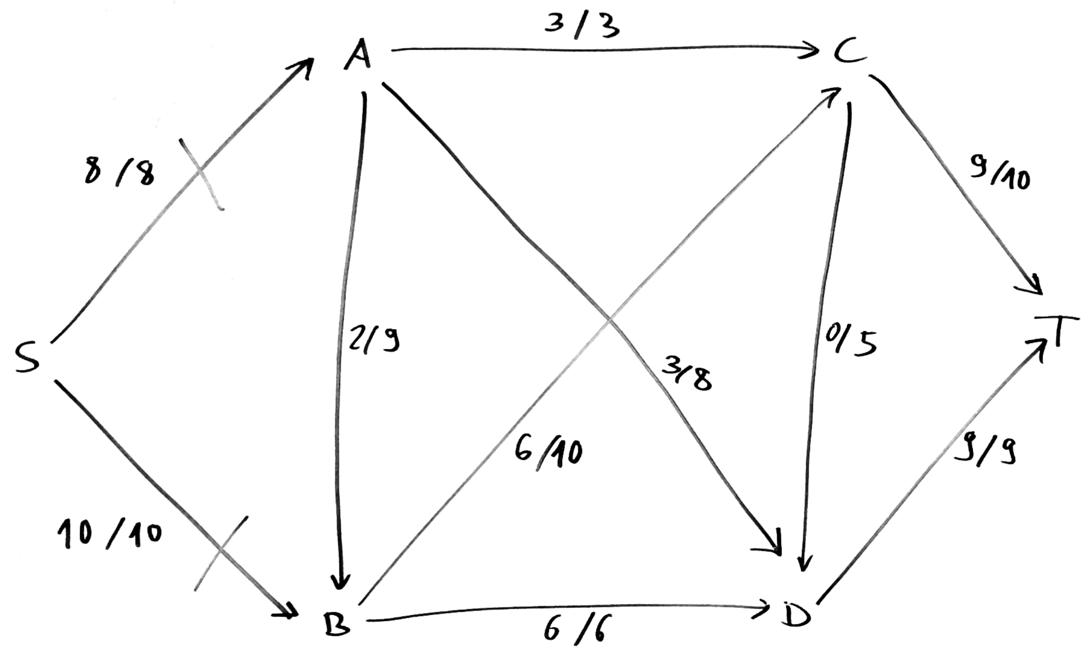
AMPL Program

```
var fSA >= 0, <= 8;
var fSB >= 0, <= 10;
var fAB >= 0, <= 9;
var fAC >= 0, <= 3;
var fBD >= 0, <= 6;
var fAD >= 0, <= 8;
var fBC >= 0, <= 6;
var fCD >= 0, <= 5;
var fCT >= 0, <= 10;
var fDT >= 0, <= 9;

maximize flow: fCT + fDT;

subject to conservationA: fSA = fAB + fAD + fAC;
subject to conservationB: fSB + fAB = fBC + fBD;
subject to conservationC: fAC + fBC = fCT + fCD;
subject to conservationD: fBD + fAD + fCD = fDT;
```

AMPL outputs the following solution with an objective function value of 18.

Edge	f_{SA}	f_{SB}	f_{AB}	f_{AC}	f_{BD}	f_{AD}	f_{BC}	f_{CD}	f_{CT}	f_{DT}
Flow	8	10	2	3	6	3	6	0	9	9

Max-flow and Min-cut



$V = A \cup B$

$C = (A, B)$, $A = \{S\}$, $B = \{A, B, C, D, T\}$

Max-Flow value = Min-Cut capacity = 18

The minimum cut cuts the edges SA and SB, resulting in two vertex sets $\{S\}$ and $V - \{S\}$.